[10]Kays, W. M., and Crawford, M. E., *Convective Heat and Mass Transfer*, 3rd ed., McGraw–Hill, New York, 1993.
[11]Weigand, B., Ferguson, J. R., and Crawford, M. E., "An Extended Kays and Crawford Turbulent Prandtl Number Model," *International Journal of Heat and Mass Transfer*, Vol. 49, No. 17, 1997, pp. 4191–4196.
[12]Modest, M. F., *Radiative Heat Transfer*, McGraw–Hill, New York, 1993.
[13]Ozisik, M. N., *Finite Difference Methods in Heat Transfer*, CRC Press, Boca Raton, FL, 1994.
[14]Hall, G., and Watt, J. M. (eds.), *Modern Numerical Methods for Ordinary Differential Equations*, Clarendon, Oxford, 1976.
[15]Kakac, S., Shah, R. K., and Aung, W., *Handbook of Single-Phase Convective Heat Transfer*, Wiley, New York, 1987.

Estimation of Wall Heat Flux in an Inverse Convection Problem

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Introduction

IN the inverse heat conduction problems, the surface conditions or the thermal properties of a material are estimated by utilizing the temperature measurements within the medium. These problems have received much attention and numerous papers have been devoted to this topic of research. Inverse radiation problems have also been investigated extensively. They are concerned with the determination of the radiative properties or the internal temperature profile of a medium from the measured radiation data. Despite the relatively large interest given in the inverse problems of heat conduction and radiation, only a small amount of work is available for the inverse heat convection problems.[1–3] In all of these studies, the unknown functions to be estimated are of one variable for the inverse convection problems. In this Note, we consider the estimation of the space- and time-dependent wall heat flux for unsteady laminar-forced convection between parallel flat plates from the temperature measurements taken inside the flow or at the opposite wall.

Analysis

Direct Problem

Consider unsteady laminar-forced convection heat transfer in a parallel plate duct with channel width b. The flow enters the channel with a fully developed velocity distribution $u(y)$ and a constant temperature T_0. Initially, the duct walls are kept thermally insulated. At time $t = 0$, the thermal condition of the upper wall at $y = b$ is suddenly changed and is subjected to wall heating condition with a function of position x and time t. The flow is assumed to have constant properties and the buoyancy term is neglected. It is intended to provide a first step toward future work, in which these effects will be considered. Figure 1 describes the geometry and coordinates. By introducing the following dimensionless quantities:

$$X = x/bPe, \qquad Y = y/b, \qquad \upsilon = \alpha t/b^2$$
$$\theta = k(T - T_0)/bq_{\text{ref}}, \qquad Pe = \bar{u}b/\alpha, \qquad U = u/\bar{u} \tag{1}$$
$$Q = q/q_{\text{ref}}, \qquad U = \tfrac{3}{2}[1 - (2Y - 1)^2]$$

where k is the thermal conductivity, α is the thermal diffusivity, T is the temperature, q is the wall heat flux, q_{ref} is the reference heat flux, and $\bar{u}$ is the mean velocity. The governing energy conservation equation in dimensionless form for the problem is given by

$$\frac{\partial \theta}{\partial \upsilon} + U\frac{\partial \theta}{\partial X} = \frac{\partial^2 \theta}{\partial Y^2} \tag{2a}$$

with the initial condition and the boundary conditions

$$\theta(X, Y, 0) = 0 \tag{2b}$$

$$\theta(0, Y, \upsilon) = 0 \tag{2c}$$

$$\frac{\partial \theta(X, 0, \upsilon)}{\partial Y} = 0 \tag{2d}$$

$$-\frac{\partial \theta(X, 1, \upsilon)}{\partial Y} = Q(X, \upsilon) \tag{2e}$$

Inverse Problem

In the direct problem, the velocity distribution, the initial condition, and the boundary conditions are given to determine the temperature distribution in the flowfield. In the inverse problem, the temperature data are assumed to be measured inside the flow or at the lower wall. The dimensionless heat flux at the upper wall, $Q(X, \upsilon)$, is recovered by using the measured data. The estimation of the wall heat flux from the knowledge of the measured temperature data can be constructed as a problem of minimization of the objective function:

$$J = \sum_{i=1}^{M} \sum_{k=1}^{N} (\theta_{i,k} - Z_{i,k})^2 \tag{3}$$

where $\theta_{i,k} = \theta(X_i, Y_1, \upsilon_k)$ is the calculated dimensionless temperature for an estimated $Q(X, \upsilon)$, and $Z_{i,k} = Z(X_i, Y_1, \upsilon_k)$ is the measured dimensionless temperature. If $Y_1 = 0$, the measurements are taken at the lower wall; if $0 < Y_1 < 1$, the measurements are taken inside the fluid. M and N are the numbers of the measured points in the X and υ directions, respectively.

In this Note, the conjugate gradient method[4] is employed to determine the unknown wall heat flux $Q(X, \upsilon)$ by minimizing the objective function J. The iterative process is

$$Q^{p+1}_{m,n} = Q^p_{m,n} - \beta^p d^p_{m,n} \tag{4}$$

where $Q_{m,n} = Q(X_m, \upsilon_n)$, β^p is the step size, and $d^p_{m,n}$ is the direction of descent, which is determined from

$$d^p_{m,n} = \left(\frac{\partial J}{\partial Q_{m,n}}\right)^p + \gamma^p d^{p-1}_{m,n} \tag{5}$$

and the conjugate coefficient γ^p is computed from

$$\gamma^p = \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\left(\frac{\partial J}{\partial Q_{m,n}}\right)^p\right]^2 \Bigg/ \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\left(\frac{\partial J}{\partial Q_{m,n}}\right)^{p-1}\right]^2$$
$$\text{with} \quad \gamma^0 = 0 \tag{6}$$

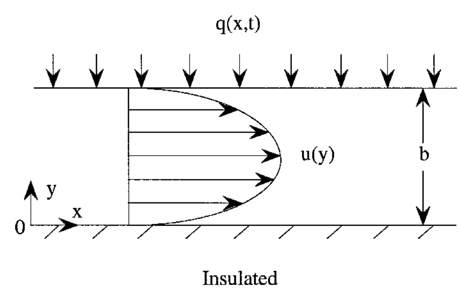

Fig. 1 Geometry and coordinates.

Received 8 July 1998; revision received 19 November 1998; accepted for publication 23 November 1998.

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Here, $\partial J / \partial Q_{m,n}$ is the gradient of the objective function. The step size is determined from

$$\beta^p = \sum_{i=1}^{M} \sum_{k=1}^{N} \left(\theta^p_{i,k} - Z_{i,k}\right) \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\partial \theta_{i,k}}{\partial Q_{m,n}}\right)^p d^p_{m,n} \Bigg/ \sum_{i=1}^{M} \sum_{k=1}^{N} \left[\sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\partial \theta_{i,k}}{\partial Q_{m,n}}\right)^p d^p_{m,n}\right]^2 \tag{7}$$

where $\partial\theta_{i,k}/\partial Q_{m,n}$ is the sensitivity coefficient. To calculate the sensitivity coefficient, the direct problem is differentiated with respect to $Q_{m,n}$ to obtain the sensitivity problem, i.e.,

$$\frac{\partial}{\partial \upsilon}\left(\frac{\partial \theta}{\partial Q_{m,n}}\right) + U \frac{\partial}{\partial X}\left(\frac{\partial \theta}{\partial Q_{m,n}}\right) = \frac{\partial^2}{\partial Y^2}\left(\frac{\partial \theta}{\partial Q_{m,n}}\right) \tag{8a}$$

$$\frac{\partial \theta(X, Y, 0)}{\partial Q_{m,n}} = 0 \tag{8b}$$

$$\frac{\partial \theta(0, Y, \upsilon)}{\partial Q_{m,n}} = 0 \tag{8c}$$

$$\frac{\partial}{\partial Y}\left[\frac{\partial \theta(X, 0, \upsilon)}{\partial Q_{m,n}}\right] = 0 \tag{8d}$$

$$-\frac{\partial}{\partial Y}\left[\frac{\partial \theta(X, 1, \upsilon)}{\partial Q_{m,n}}\right] = \hat{u}(X - X_m, \upsilon - \upsilon_n) \tag{8e}$$

for $m = 1, 2, \ldots, M$; $n = 1, 2, \ldots, N$, where

$$\hat{u}(X - X_m, \upsilon - \upsilon_n) = \begin{cases} 1 & \text{if} \quad X = X_m,\ \upsilon = \upsilon_n \\ 0 & \text{otherwise} \end{cases} \tag{8f}$$

The gradient of the objective function $\partial J / \partial Q_{m,n}$ is determined by differentiating Eq. (3) with respect to $Q_{m,n}$ to obtain

$$\frac{\partial J}{\partial Q_{m,n}} = 2 \sum_{i=1}^{M} \sum_{k=1}^{N} (\theta_{i,k} - Z_{i,k}) \frac{\partial \theta_{i,k}}{\partial Q_{m,n}} \tag{9}$$

If the problem contains no measurement errors, the condition

$$J\left(Q^p_{m,n}\right) < \delta \tag{10}$$

can be used for terminating the iterative process, where δ is a small specified positive number. However, the measured temperature data contain measurement errors. Following the computational experience, we use the discrepancy principle[5]:

$$J\left(Q^p_{m,n}\right) < MN\sigma^2 \tag{11}$$

as the stopping criterion, where σ is the standard deviation of the measurement errors.

The computational procedure for the solution of the inverse convection problem is summarized as follows:

Step 1: Solve the sensitivity problem to calculate the sensitivity coefficient $\partial\theta_{i,k}/\partial Q_{m,n}$.

Step 2: Pick an initial guess $Q^0_{m,n}$. Set $p = 0$.

Step 3: Solve the direct problem to compute $\theta_{i,k}$.

Step 4: Calculate the objective function. Terminate the iteration process if the specified stopping criterion is satisfied. Otherwise go to step 5.

Step 5: Knowing $\partial\theta_{i,k}/\partial Q_{m,n}$, $\theta_{i,k}$, and $Z_{i,k}$, compute the gradient of the objective function $\partial J / \partial Q_{m,n}$.

Step 6: Knowing $\partial J / \partial Q_{m,n}$, compute γ^p and $d^p_{m,n}$.

Step 7: Knowing $\partial\theta_{i,k}/\partial Q_{m,n}$, $\theta_{i,k}$, $Z_{i,k}$, and $d^p_{m,n}$, compute β^p.

Step 8: Knowing β^p and $d^p_{m,n}$, compute $Q^{p+1}_{m,n}$. Set $p = p + 1$ and go to step 3.

Results and Discussion

Two test cases are used to demonstrate the accuracy of the proposed method for the estimation of the upper wall heat flux from the simulated measured temperature data. The effects of the measurement error, the sensor location, and the functional form of the wall heat flux on the results of the inverse analysis are investigated. The measured temperature data Z are simulated by adding random errors to the exact temperature θ computed from the solution of the direct problem:

$$Z = \theta + \sigma\zeta \tag{12}$$

where σ is the standard deviation of the measurement data, and ζ is a random variable of normal distribution with zero mean and unit standard deviation. Forty-one equally spaced measurements are taken both in $0 \le X \le 1$ and $0 \le \upsilon \le 1$ for all of the cases considered in this work. The data are used as input to reconstruct the unknown wall heat flux in the inverse problem.

In the first case, the unknown wall heat flux is assumed to be a function of υ only:

$$Q(X, \upsilon) = \begin{cases} 20\upsilon & 0 \le \upsilon \le 0.5 \\ 20(1 - \upsilon) & 0.5 \le \upsilon \le 1 \end{cases} \tag{13}$$

We consider the inverse solutions for the temperature measurements taken inside the fluid or at the lower wall. The results show that the closer to the unknown wall heat flux the sensors are, the more accurate the estimation. The solutions of the inverse analysis from

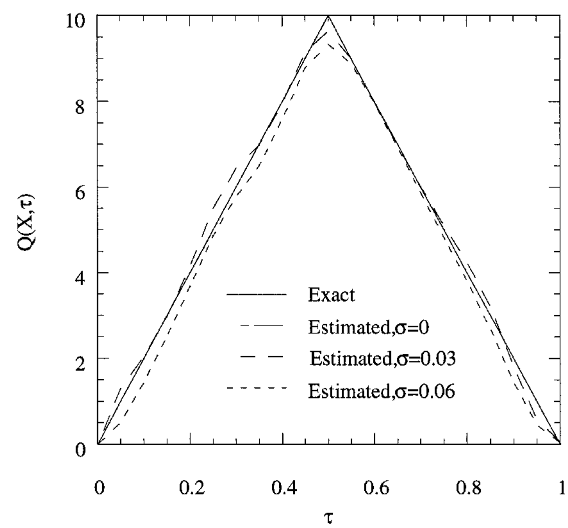

Fig. 2 Estimation of the wall heat flux by inverse analysis, $Y_1 = 0$.

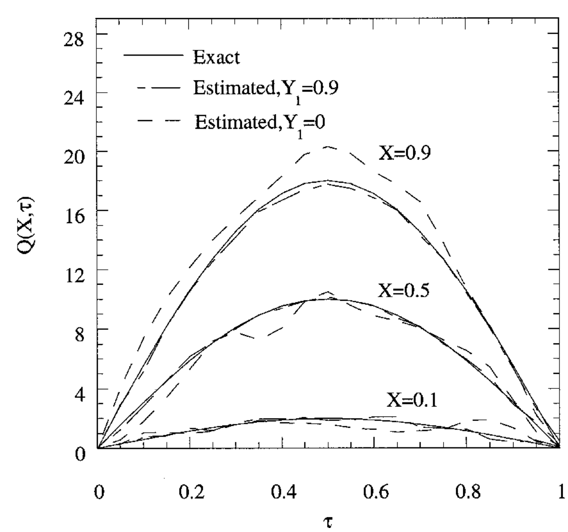

Fig. 3 Estimation of the wall heat flux by inverse analysis, $\sigma = 0.03$.

temperature measurements taken at the lower wall for both exact and noisy input data are shown in Fig. 2. The estimation of the inverse problem is excellent for exact input data, i.e., $\sigma = 0$. The accuracy of the inverse analysis is also good for simulated experimental data containing errors of standard deviation $\sigma = 0.03$ and 0.06, which correspond to approximate 2 and 4% errors of the maximum measured dimensionless temperature, respectively. As σ is increased, the accuracy of the estimation decreases.

In the second case, the unknown wall heat flux is assumed to be a function of both X and υ:

$$Q(X, \upsilon) = 20X \sin(\pi \upsilon) \tag{14}$$

Figure 3 is used to demonstrate the effects of the locations of measurements on the accuracy of the inverse analysis. When the measurements are inside the fluid, i.e., $Y_1 = 0.9$, the agreement between the estimated and the exact values of the wall heat flux is good. The result of the inverse analysis is satisfactory when the measured data are taken at the lower wall, i.e., $Y_1 = 0$. As expected, the accuracy of the estimation is improved when the distance between the upper wall and the sensors is decreased. However, from the experimental point of view, it is desirable to avoid sensors within the fluid that will disturb the flowfield and introduce errors.

Conclusions

The estimation of the space- and time-dependent wall heat flux for unsteady laminar-forced convection between flat plates has been considered. The conjugate gradient method is applied to solve the problem. Various types of wall heat fluxes are used to test the accuracy of the method. The inverse solutions are satisfactory for both exact and noisy data. As expected, the results also show that the closer to the unknown wall heat flux the sensors are, the more accurate the estimation.

Acknowledgment

The support of this work by the National Science Council of the Republic of China under Contract NSC 88-2212-E-211-003 is gratefully acknowledged.

References

[1] Moutsoglou, A., "Solution of an Elliptic Inverse Convection Problem Using a Whole Domain Regularization Technique," *Journal of Thermophysics and Heat Transfer*, Vol. 4, No. 3, 1990, pp. 341–349.

[2] Huang, C. H., and Ozisik, M. N., "Inverse Problem of Determining Unknown Wall Heat Flux in Laminar Flow Through a Parallel Plate Duct," *Numerical Heat Transfer, Part A*, Vol. 21, No. 1, 1992, pp. 101–116.

[3] Liu, F. B., and Ozisik, M. N., "Inverse Analysis of Transient Turbulent Forced Convection Inside Parallel-Plate Ducts," *International Journal of Heat and Mass Transfer*, Vol. 39, No. 12, 1996, pp. 2615–2618.

[4] Hestenes, M. R., *Conjugate Direction Methods in Optimization*, Springer–Verlag, New York, 1980, Chap. 4.

[5] Alifanov, O. M., "Solution of an Inverse Problem of Heat Conduction by Iteration Methods," *Journal of Engineering Physics*, Vol. 26, No. 4, 1974, pp. 471–476.